

Numerical Ranges over Finite Fields

Let p be a prime number with remainder 3 after dividing by 4. The field \mathbf{Z} is system in which only the integers zero through $p-1$ exist. The field $\mathbf{F}=\mathbf{Z}[i]$ is the field \mathbf{Z} augmented with i , the square root of $p-1$. Elements of \mathbf{F} are of the form $a+bi$, where a and b are elements of \mathbf{Z} .

Pick a number k in \mathbf{Z} , and let S be the set of all vectors x with entries in \mathbf{F} where the product $T(x)x=k$, and $T(x)$ is the conjugate transpose of x . The author has created a definition of a new concept, the k -numerical range $W(A, k)$, which is the set of numbers of the form $T(x)Ax$, for all x in S . We investigate the properties of these k -numerical ranges, and explore the fundamental differences between $W(A, 0)$ and $W(A, k)$ for nonzero k . We will then discuss our pioneering work in classifying the shapes which $W(A, 1)$ can take. This includes a new proof of the existence of A for which $W(A, 1)$ is a union of pairwise disjoint lines or $W(A, 1)$ is a single line through the origin. This investigation embodies a common practice in mathematics to take a topic that lives in a continuous system, in this case the numerical range over the standard complex numbers, and look for and examine an analog in a discrete system. Doing so can provide new insights and lead to hints about how to solve larger problems in the continuous system. This is what we hope to accomplish by beginning the investigation of k -numerical ranges over finite fields.

Key Words: numerical ranges, matrices, linear algebra, mathematics, field of values, eigenvalue